1. Angular Momentum in Quantum Mechanics

Before starting a detailed discussion of the underlying mechanisms that establish the nuclear shell model, not only for the single-particle degrees of freedom and excitations but also in order to study nuclei where a number of valence nucleons (protons and/or neutrons) are present outside closed shells, the quantum mechanical methods are discussed in some detail. We discuss both angular momentum in the framework of quantum mechanics and the aspect of rotations in quantum mechanics with some side-steps to elements of groups of transformations. Although in these Chaps. 1 and 2, not everything is proved, it should supply all necessary tools to tackle the nuclear shell model with success and with enough background to feel at ease when manipulating the necessary "Racah"-algebra (Racah 1942a, 1942b, 1943, 1949, 1951) needed to gain better insight into how the nuclear shell model actually works. These two chapters are relatively self-contained so that one can work through them without constant referral to the extensive literature on angular momentum algebra. More detailed discussions are in the appendices. We also include a short summary of often used expressions for the later chapters.

1.1 Central Force Problem and Orbital Angular Momentum

A classical particle, moving in a central one-particle field $U(r)$ can be described by the single-particle Hamiltonian (Brussaard, Glaudemans 1977)

$$H = \frac{\boldsymbol{p}^2}{2m} + U(r) \,. \tag{1.1}$$

In quantum mechanics, since the linear momentum $\boldsymbol{p}$ has to be replaced by the operator $-\mathrm{i}\,\hbar\boldsymbol{\nabla}$, this Hamiltonian becomes

$$H = -\frac{\hbar^2}{2m}\Delta + U(r) \,, \tag{1.2}$$

where Δ is the Laplacian operator. The orbital angular momentum itself is defined as

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} \,, \qquad \text{or} \tag{1.3}$$

$$\boldsymbol{l} = -\mathrm{i}\,\hbar\boldsymbol{r} \times \boldsymbol{\nabla} \,, \tag{1.4}$$

as the corresponding quantum mechanical angular momentum operator. The components can be easily obtained in an explicit way by using the determinant notation for $\boldsymbol{l}$, i.e.,

$$-\mathrm{i}\,\hbar \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{vmatrix} , \tag{1.5}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote unit vectors in the x, y and z direction, respectively.

The commutation rules between the different components of the angular momentum operator can be easily calculated using the relations

$$\left[x,\, p_x\right] = x p_x - p_x x = \mathrm{i}\,\hbar \,, \tag{1.6}$$

which leads to the results

$$\left[l_x,\, l_y\right] = \mathrm{i}\,\hbar l_z \,, \tag{1.7}$$

with cyclic permutations.

We can furthermore define the operator that expresses the total length of the angular momentum as

$$\boldsymbol{l}^2 = l_x^2 + l_y^2 + l_z^2 \,, \tag{1.8}$$

which has the following commutation relations with the separate components

$$\left[\boldsymbol{l}^2,\, l_i\right] = 0 \qquad (i \equiv x,\, y,\, z) \,. \tag{1.9}$$

If we now try to determine the one-particle Schrödinger equation that corresponds to the central force problem of (1.1), we can use a shorthand method for evaluating the operator $\boldsymbol{l}^2$. Starting from the commutation relations (1.6), one can show that

$$\boldsymbol{l}^2 = (\boldsymbol{r} \times \boldsymbol{p}) \cdot (\boldsymbol{r} \times \boldsymbol{p}) = r^2 p^2 - \boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{p}) \cdot \boldsymbol{p} + 2\mathrm{i}\,\hbar \boldsymbol{r} \cdot \boldsymbol{p} \,, \tag{1.10}$$

and using

$$\boldsymbol{r} \cdot \boldsymbol{p} = -\mathrm{i}\,\hbar r \frac{\partial}{\partial r} \,, \tag{1.11}$$

one obtains

$$\boldsymbol{l}^2 = \boldsymbol{r}^2 \boldsymbol{p}^2 + \hbar^2 \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) . \tag{1.12}$$

The kinetic energy operator of (1.1) then becomes

$$\boldsymbol{T} = \frac{\boldsymbol{p}^2}{2m} \doteq \frac{\boldsymbol{l}^2}{2mr^2} - \frac{\hbar^2}{2mr^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) . \tag{1.13}$$

We shall now briefly recapitulate the solutions to the central one-body Schrödinger equation, solutions that form a basis of eigenfunctions of the operators $\boldsymbol{H}$, $\boldsymbol{l}^2$ and l_z simultaneously. So, we can write still in a rather general way that

$$H\varphi(\boldsymbol{r}) = E\varphi(\boldsymbol{r}) , \tag{1.14}$$

$$\boldsymbol{l}^2\varphi(\boldsymbol{r}) = \hbar^2\lambda\varphi(\boldsymbol{r}) . \tag{1.15}$$

The Schrödinger equation (1.14) now becomes [using (1.15)]

$$\left[-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\lambda\hbar^2}{2mr^2} + U(r)\right]\varphi(\boldsymbol{r}) = E\varphi(\boldsymbol{r}) . \tag{1.16}$$

Using a separable solution of the type

$$\varphi(\boldsymbol{r}) \equiv R(r)Y(\theta,\varphi) = \frac{u(r)}{r}\cdot Y(\theta,\varphi) , \tag{1.17}$$

the radial equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left[\frac{\lambda\hbar^2}{2mr^2} + U(r)\right]u(r) = Eu(r) , \tag{1.18}$$

and its solution, in particular, depends on the choice of the form of the central potential $U(r)$. This particular problem will be discussed in Chap. 3.

The eigenfunctions for the angular part of (1.16) can be obtained most easily by rewriting the angular momentum $\boldsymbol{l}^2$ operator explicitly in a basis of spherical coordinates (Fig. 1.1). One works as follows:

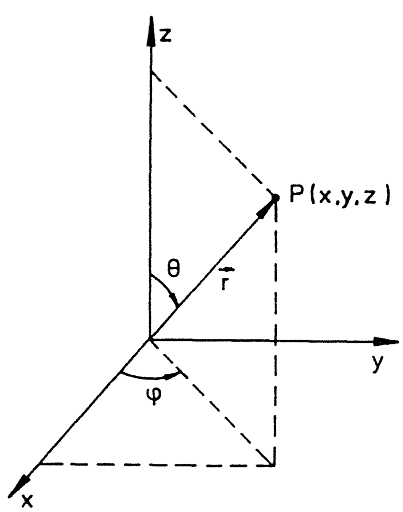


Fig. 1.1. The cartesian and spherical coordinates for the point $P(\boldsymbol{r})$ (x, y, z) and (r, θ, φ) for which the orbital angular momentum is analyzed

i) rewrite the cartesian components l_x, l_y, l_z as a function of the spherical coordinates (r, θ, φ)

$$\begin{aligned} l_x &= \mathrm{i}\,\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right), \\ l_y &= \mathrm{i}\,\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right), \\ l_z &= -\mathrm{i}\,\hbar\frac{\partial}{\partial\varphi}\,. \end{aligned} \tag{1.19}$$

For more details, see (Sect. 1.3).

ii) rewrite the length of the angular momentum $\boldsymbol{l}^2$ as a function of the spherical coordinates

$$\begin{aligned} \boldsymbol{l}^2 &= l_x^2 + l_y^2 + l_z^2 \\ &= -\hbar^2\left\{\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\right. \\ &\quad + \left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \\ &\quad \times\left.\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{\partial^2}{\partial\varphi^2}\right\} \\ &= -\hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}. \end{aligned} \tag{1.20}$$

We now determine the angular momentum eigenfunctions starting from (1.15, 20). Using a separable form

$$Y(\theta,\varphi) = \Phi(\varphi)\cdot\Theta(\theta)\,, \tag{1.21}$$

one gets the two differential equations

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0\,, \tag{1.22}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right) - \frac{m^2}{\sin^2\theta}\Theta + \lambda\Theta = 0\,. \tag{1.23}$$

In order to obtain these two equations, one uses a separation method for the variables θ and φ as is discussed in introductory courses on quantum mechanics (Flügge 1974). The solutions to (1.22), using the condition of uniqueness of the solutions, become

$$\Phi(\varphi) = \mathrm{e}^{\mathrm{i}m\varphi} \qquad m = 0,\ \pm1,\ \pm2,\ \ldots\,. \tag{1.24}$$

Putting now $\lambda = l(l+1)$ and $\xi = \cos\theta$, one recognizes in (1.23) the differential equation for the associated Legendre polynomials P_l^m (Edmonds 1957), i.e., one has $(0 \le |m| \le l)$

$$P_l^m(\xi) = \left(1-\xi^2\right)^{m/2} \frac{d^m}{d\xi^m} P_l(\xi) , \tag{1.25}$$

with

$$P_l(\xi) = \frac{1}{2^l l!} \frac{d^l}{d\xi^l} \left(\xi^2 - 1\right)^l . \tag{1.26}$$

(P_l are the Legendre polynomials).

Finally, the solution to (1.15) becomes

$$Y_l^m(\theta,\varphi) = \sqrt{\frac{(2l+1)}{4\pi}\left(\frac{(l-m)!}{(l+m)!}\right)}(-1)^m \, \mathrm{e}^{im\varphi} \, P_l^m(\cos\theta) , \tag{1.27}$$

with correct angular normalization, and using $m \ge 0$. For negative values of m, one has the relation

$$Y_l^{-m}(\theta,\varphi) = (-1)^m Y_l^m(\theta,\varphi)^* .$$

These functions are well known as the spherical harmonics. Using the above solutions, the angular momentum eigenvalue equations can be written

$$l^2 Y_l^m(\theta,\varphi) = \hbar^2 l(l+1) Y_l^m(\theta,\varphi) , \tag{1.28}$$

$$l_z Y_l^m(\theta,\varphi) = \hbar m Y_l^m(\theta,\varphi) . \tag{1.29}$$

We give here some of the most used spherical harmonics.

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} , \tag{1.30}$$

$$\begin{aligned} Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos\theta , \\ Y_1^{\pm1} &= \mp\sqrt{\frac{3}{8\pi}} \mathrm{e}^{\pm i\varphi} \sin\theta , \end{aligned} \tag{1.31}$$

$$\begin{aligned} Y_2^0 &= \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right) , \\ Y_2^{\pm1} &= \mp\sqrt{\frac{15}{8\pi}} \mathrm{e}^{\pm i\varphi} \cos\theta \sin\theta , \\ Y_2^{\pm2} &= \sqrt{\frac{15}{32\pi}} \mathrm{e}^{\pm 2i\varphi} \sin^2\theta . \end{aligned} \tag{1.32}$$

In Fig. 1.2, we illustrate some typical linear combinations, which are called the s, p and d functions (Weissbluth 1974). These functions play a major role when describing electronic bonds in molecule formation. These are the combinations

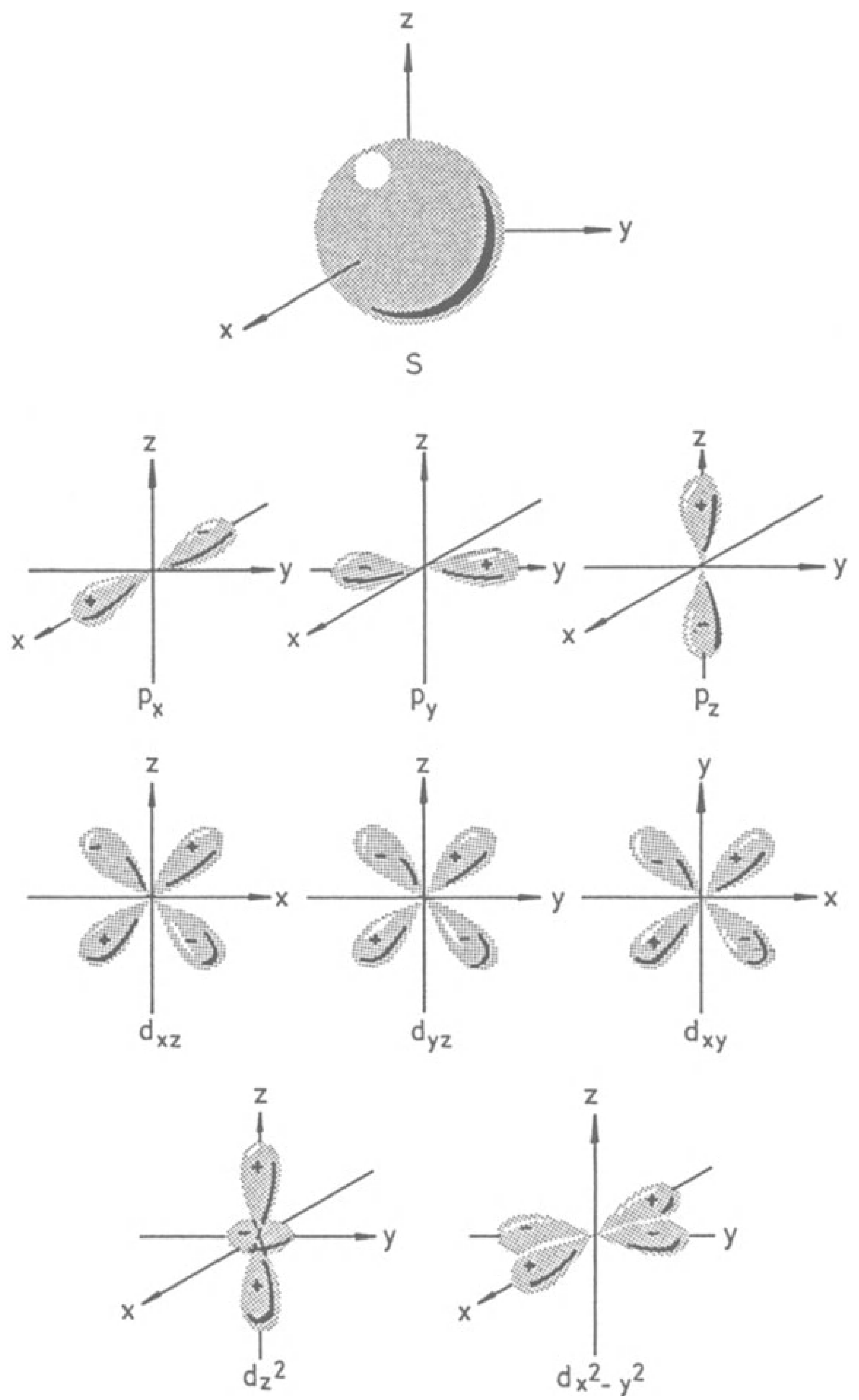


Fig. 1.2. Polar diagrams for s ($l = 0$), p ($l = 1$) and d ($l = 2$) angular wave functions. These represent real combinations of the Y_0^0, Y_1^μ and Y_2^μ spherical harmonics. The figure is taken from C.J. Ballhausen and H.B. Gray "Molecular Orbital Theory", W.A. Benjamin, Inc.

$$
\begin{aligned}
s &= \sqrt{4\pi} Y_0^0 \ , \\
p_x &= \sqrt{\frac{4\pi}{3}} \sqrt{\frac{1}{2}} \left(-Y_1^1 + Y_1^{-1} \right) r = x \ , \\
p_y &= \sqrt{\frac{4\pi}{3}} \sqrt{\frac{1}{2}} i \left(Y_1^1 + Y_1^{-1} \right) r = y \ , \\
p_z &= \sqrt{\frac{4\pi}{3}} Y_1^0 \cdot r = z \ , \\
d_{z^2} &= \sqrt{\frac{4\pi}{5}} Y_2^0 \cdot r^2 = \frac{1}{2} \left(3z^2 - r^2 \right) \ , \\
d_{x^2y^2} &= \sqrt{\frac{4\pi}{5}} \sqrt{\frac{1}{2}} \left(Y_2^2 + Y_2^{-2} \right) \cdot r^2 = \frac{1}{2} \sqrt{3} \left(x^2 - y^2 \right) \ , \\
d_{xy} &= \sqrt{\frac{4\pi}{5}} \sqrt{\frac{1}{2}} \mathrm{i} \left(-Y_2^2 + Y_2^{-2} \right) \cdot r^2 = \sqrt{3} xy \ , \\
d_{yz} &= \sqrt{\frac{4\pi}{5}} \sqrt{\frac{1}{2}} \mathrm{i} \left(Y_2^1 + Y_2^{-1} \right) \cdot r^2 = \sqrt{3} yz \ , \\
d_{zx} &= \sqrt{\frac{4\pi}{5}} \sqrt{\frac{1}{2}} \left(-Y_2^1 + Y_2^{-1} \right) \cdot r^2 = \sqrt{3} zx \ .
\end{aligned}
\tag{1.33}
$$

The angular momentum eigenfunctions are a set of orthogonal functions on the unit sphere expressed by

$$
\int_0^{2\pi} \int_0^{\pi} Y_l^{m*}(\theta, \varphi) Y_{l'}^{m'}(\theta, \varphi) \sin\theta \, d\theta \, d\varphi = \delta_{ll'} \delta_{mm'} \ . \tag{1.34}
$$

Also, an addition theorem exists

$$
P_l(\cos\theta_{12}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} Y_l^{m*}(\Omega_1) Y_l^m(\Omega_2) \ , \tag{1.35}
$$

where Ω_1, Ω_2 are the angles (θ_1,φ_1), (θ_2,φ_2) defining the two directions and θ_{12} is the angle between the two direction vectors $\boldsymbol{\Omega}_1$,$\boldsymbol{\Omega}_2$.

We now introduce angular momentum ladder operators $l\pm$, operators that are linear combinations of the operators l_x, l_y but are very useful in setting up the angular momentum algebra relations. Defining

$$
l_\pm \equiv l_x \pm \mathrm{i} l_y \ , \tag{1.36}
$$

we shall determine the action of these ladder operators acting on the spherical harmonics. Therefore, we just need to evaluate the commutation relations of the ladder operators among themselves and with l_z. One can easily evaluate that

$$
\left[l_+, l_- \right] = 2\hbar l_z \ , \quad \left[l_z, l_+ \right] = \hbar l_+ \ , \quad \left[l_z, l_- \right] = -\hbar l_- \ . \tag{1.37}
$$

Using the spherical coordinates and the explicit forms of l_x, l_y, l_z (1.17–19), one can rewrite l_+, l_- and l_z as

$$\begin{aligned} l_+ &= \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \\ l_- &= -\hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) \\ l_z &= -i\hbar\frac{\partial}{\partial\varphi}\,. \end{aligned} \tag{1.38}$$

Knowing the explicit form of the spherical harmonics $Y_l^m(\theta,\varphi)$, the action of the operators $l\pm$, l_z on these functions can, in principle, be calculated in a straightforward but tedious way. We shall discuss a more elegant method in evaluating the action of $l\pm$ on the spherical harmonics. We start from the eigenvalue equation (1.29) on which we act with the operator l_+ giving

$$l_+l_zY_l^m(\theta,\varphi) = m\hbar l_+Y_l^m(\theta,\varphi)\,. \tag{1.39}$$

Now, using the commutation relations (1.37), this relation can be rewritten as

$$l_+l_zY_l^m(\theta,\varphi) = \left(l_z - \hbar\right)l_+Y_l^m(\theta,\varphi) = m\hbar l_+Y_l^m(\theta,\varphi)\,, \tag{1.40}$$

or,

$$l_z\left(l_+Y_l^m(\theta,\varphi)\right) = (m+1)\hbar\left(l_+Y_l^m(\theta,\varphi)\right)\,. \tag{1.41}$$

This indicates that $l_+Y_l^m(\theta,\varphi)$ is an eigenfunction of the operator l_z with eigenvalue $(m+1)\hbar$ and thus the ladder operator l_+ effectively adds one unit $\hbar$ to the original m-projection. Likewise, l_- subtracts one unit $\hbar$ from the original m-projection and gives

$$l_z\left(l_-Y_l^m(\theta,\varphi)\right) = (m-1)\hbar\left(l_-Y_l^m(\theta,\varphi)\right)\,. \tag{1.42}$$

This allows for the relations

$$\begin{aligned} l_+Y_l^m(\theta,\varphi) &= \alpha(l,m)Y_l^{m+1}(\theta,\varphi)\,, \\ l_-Y_l^m(\theta,\varphi) &= \beta(l,m)Y_l^{m-1}(\theta,\varphi)\,. \end{aligned} \tag{1.43}$$

The factors $\alpha(l,m)$, $\beta(l,m)$ can be determined by calculating the norm of expressions (1.43). Thus,

$$\int Y_l^{m*}(\theta,\varphi)l_-l_+Y_l^m(\theta,\varphi)d\Omega = |\alpha(l,m)|^2\,, \tag{1.44}$$

since the spherical harmonics form an orthonormal set of eigenfunctions. We can now evaluate the operator expression l_-l_+ explicitly as follows. We start from

$$l^2 = l_x^2 + l_y^2 + l_z^2 = \tfrac{1}{2}\left(l_+l_- + l_-l_+\right) + l_z^2\,, \tag{1.45}$$

and using the commutation relations (1.37), this simplifies into

$$l^2 = l_-l_+ + l_z\left(l_z + \hbar\right)\,, \tag{1.46}$$

giving rise to the equality

$$l_- l_+ = \boldsymbol{l}^2 - l_z(l_z + \hbar) \ . \tag{1.47}$$

Here, one also needs to impose the conditions

$$l_+ Y_l^l = l_- Y_l^{-l} = 0 \ .$$

This relation (1.47) used in (1.44) gives the result

$$\hbar^2[l(l+1) - m(m+1)] = |\alpha(l,m)|^2 \ , \tag{1.48}$$

or

$$l_+ Y_l^m(\theta,\varphi) = \hbar\{l(l+1) - m(m+1)\}^{1/2} Y_l^{m+1}(\theta,\varphi) \ . \tag{1.49}$$

Similarly, for the other ladder operator l_- one has

$$l_- Y_l^m(\theta,\varphi) = \hbar\{l(l+1) - m(m-1)\}^{1/2} Y_l^{m-1}(\theta,\varphi) \ . \tag{1.50}$$

1.2 General Definitions of Angular Momentum

In Sect. 1.1, we have derived the angular momentum operator $\boldsymbol{l}(l_x, l_y, l_z)$ explicitly, starting from the one-body central force problem. This method only allows for entire values of the angular momentum eigenvalue l and m. It is now possible to define angular momentum in a more general but abstract way starting from the commutation rules (1.7,9). If we construct general operators $\boldsymbol{J}^2$, $J_i (i \equiv x,y,z)$ which fulfill the relations

$$\begin{aligned} \left[\boldsymbol{J}^2, J_i\right] &= 0 \qquad (i \equiv x,y,z) \ , \\ \left[J_x, J_y\right] &= \mathrm{i}\,\hbar J_z \ , \end{aligned} \tag{1.51}$$

and cyclic permutations, the operator $\boldsymbol{J}^2$ defines a general angular momentum operator. The eigenvectors are now defined as the abstract vectors in a Hilbert space carrying two quantum numbers, i.e., the quantum number defining the length j and the quantum number defining the projection of $j(m)$, since the quantum numbers corresponding to the full set of commuting operators define the state vector uniquely. Thus, we have the eigenvector relations

$$\boldsymbol{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \ , \tag{1.52}$$

$$J_z|j,m\rangle = \hbar m|j,m\rangle \ . \tag{1.53}$$

Using the ladder operators, we can also write

$$J_\pm|j,m\rangle = \hbar\{j(j+1) - m(m\pm 1)\}^{1/2}|j,m\pm 1\rangle \ , \tag{1.54}$$

$$J_z|j,m\rangle = \hbar m|j,m\rangle \ , \tag{1.55}$$

as the defining expressions for a general angular momentum operator.

1.2.1 Matrix Representations

In the discussion of Sect. 1.1, the angular momentum operators had an explicit expression in terms of the coordinates and derivatives to these coordinates (differential form). In the more general case, as discussed above, we can derive a matrix representation of the operators J_x, J_y, J_z or J_+, J_-, J_z and $\boldsymbol{J}^2$ within the space spanned by the state vectors $|j,m\rangle$. As an example, we use the five state vectors $|j,m\rangle$ for the case of $j = 2(-2 \leq m \leq +2)$. We denote the state vectors in column vector form as

$$\begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} , \quad \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} , \ldots , \quad \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} , \tag{1.56}$$

$$|j,2\rangle , \quad |j,1\rangle , \ldots , \quad |j,-2\rangle .$$

The action of the ladder operators now leads to

$$J_+|j,m\rangle = a_{\bar{m},\bar{m}+1}|j,m+1\rangle , \tag{1.57}$$

where $a_{\bar{m},\bar{m}+1}$ defines the following matrix representation of J_+, i.e.,

$$J_+ \Rightarrow \begin{vmatrix} 0 & a_{1,2} & 0 & 0 & 0 \\ 0 & 0 & a_{2,3} & 0 & 0 \\ 0 & 0 & 0 & a_{3,4} & 0 \\ 0 & 0 & 0 & 0 & a_{4,5} \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} . \tag{1.58}$$

Similarly, one gets a matrix representation for J_- following from

$$J_-|j,m\rangle = b_{\bar{m},\bar{m}-1}|j,m-1\rangle , \tag{1.59}$$

and for J_z since

$$J_z|j,m\rangle = \hbar m \delta_{m,m'}|j,m\rangle . \tag{1.60}$$

1.2.2 Example for Spin $\frac{1}{2}$ Particles

The angular momentum representation for spin $\frac{1}{2}$ particles (electron, proton, ...), using the general method as outlined in Sect. 1.2, now gives in a simple application the construction of the 2×2 spin matrices. We briefly recapitulate the spin $\frac{1}{2}$ angular momentum commutation relations.

$$\left[s_+, s_-\right] = 2\hbar s_z , \quad \left[s_z, s_+\right] = \hbar s_+ , \quad \left[s_z, s_-\right] = -\hbar s_- . \tag{1.61}$$

Defining $\boldsymbol{s} = \hbar/2\boldsymbol{\sigma}$, these commutation relations become

$$\left[\sigma_+, \sigma_-\right] = 4\sigma_z , \quad \left[\sigma_z, \sigma_+\right] = 2\sigma_+ , \quad \left[\sigma_z, \sigma_-\right] = -2\sigma_- . \tag{1.62}$$

The matrix representations are spanned in the two-dimensional space defined by the state vectors

$$\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix}, \tag{1.63}$$

which correspond to the states $|\frac{1}{2}, +\frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$, respectively. One often denotes the former by $\chi^{1/2}_{+1/2}$ or $\alpha(s)$ and the latter by $\chi^{1/2}_{-1/2}$ or $\beta(s)$ in literature on angular momentum (Edmonds 1957, de-Shalit, Talmi 1963, Rose, Brink 1967, Brussaard, Glaudemans 1977). The ladder operator relations (1.54) for the specific case of spin 1/2 particles become

$$\begin{aligned} s_+|\tfrac{1}{2}, -\tfrac{1}{2}\rangle &= \hbar|\tfrac{1}{2}, +\tfrac{1}{2}\rangle , \\ s_-|\tfrac{1}{2}, +\tfrac{1}{2}\rangle &= \hbar|\tfrac{1}{2}, -\tfrac{1}{2}\rangle , \end{aligned} \tag{1.64}$$

or

$$\begin{aligned} \sigma_+|\tfrac{1}{2}, -\tfrac{1}{2}\rangle &= 2|\tfrac{1}{2}, +\tfrac{1}{2}\rangle , \\ \sigma_-|\tfrac{1}{2}, +\tfrac{1}{2}\rangle &= 2|\tfrac{1}{2}, -\tfrac{1}{2}\rangle . \end{aligned} \tag{1.65}$$

One immediately gets the σ_x, σ_y and σ_z “operators” as

$$\sigma_x \equiv \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}. \tag{1.66}$$

Finally, we give a number of interesting properties for the Pauli spin $\frac{1}{2}$ matrices without proof:

$$\text{i) } \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1} , \tag{1.67}$$

where $\mathbb{1}$ denotes the 2×2 unit matrix.

$$\text{ii) } \left\{\sigma_x, \sigma_y\right\} = 0 , \tag{1.68}$$

and cyclic where $\{A, B\}$ is the anti-commutator defined as $AB + BA$.

$$\text{iii) } (\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{B} + \mathrm{i}\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}) , \tag{1.69}$$

if both $\boldsymbol{A}$ and $\boldsymbol{B}$ commute with $\boldsymbol{\sigma}$.

1.3 Total Angular Momentum for a Spin $\frac{1}{2}$ Particle

The total wave function characterizing a particle with intrinsic spin $\frac{1}{2}$ (electron, proton, neutron, ...) which, at the same time, carries orbital angular momentum can be written as the product wave function

$$\begin{aligned} \varphi(\boldsymbol{r}, \boldsymbol{\sigma}) &= \psi\left(lm_l, \tfrac{1}{2}m_s\right) \\ &= R_{nl}(r)Y_l^{m_l}(\theta, \varphi)\chi^{1/2}_{m_s}(\boldsymbol{\sigma}) , \end{aligned} \tag{1.70}$$

where $R_{n,l}(r)$ describes the solution of the radial Schrödinger equation (1.18) using a given potential $U(r)$, n describes a radial quantum number counting the number of nodes, and l the orbital angular momentum eigenvalue. Furthermore, $Y_l^{m_l}(\theta, \varphi)$ (with $-l \leq m_l \leq +l$) describes the angular part of the wave function and $\chi_{m_s}^{1/2}(\sigma)$ ($m_s = \pm\frac{1}{2}$) the intrinsic spin wave function. Since $\chi_{m_s}^{1/2}$ can be written as a state vector in a two-dimensional space, it is more correct to speak of (1.70) as a state vector than as a wave function.

The following eigenvalue equations are fulfilled for the state vectors (1.70):

$$\begin{aligned} \boldsymbol{l}^2\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) &= \hbar^2 l(l+1)\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) , \\ l_z\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) &= \hbar m_l\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) , \\ \boldsymbol{s}^2\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) &= \hbar^2\tfrac{3}{4}\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) , \\ s_z\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) &= \hbar m_s\psi\left(lm_l,\, \tfrac{1}{2}m_s\right) . \end{aligned} \tag{1.71}$$

We now define the operator

$$\boldsymbol{J} = \boldsymbol{l} + \boldsymbol{s} = \boldsymbol{l} + \hbar/2\boldsymbol{\sigma} . \tag{1.72}$$

Using the definitions for a general angular momentum operator (1.51), one can show that the operator $\boldsymbol{J}(\boldsymbol{J}^2,\, J_x,\, J_y,\, J_z)$ is indeed an angular momentum operator since the commutation relations

$$\begin{aligned} \left[J_x,\, J_y\right] &= \mathrm{i}\,\hbar J_z \;, \ldots \;, \\ \left[\boldsymbol{J}^2,\, J_i\right] &= 0 \;, \end{aligned} \tag{1.73}$$

hold (verify this explicitly).

By construction $\boldsymbol{l}^2$, l_z, $\boldsymbol{s}^2$ and s_z form a set of commuting operators (the orbital angular momentum operators and the intrinsic angular momentum "spin" operators act in totally different spaces, the former in the space of coordinates (x, y, z), the latter in an abstract space, spanned by the unit vectors $\chi_{m_s}^{1/2}$).

One can now show that the operators $\boldsymbol{J}^2$, J_z, $\boldsymbol{l}^2$ and $\boldsymbol{s}^2$ also form a set of commuting operators indicating that it is also possible to describe the full state of the particle with orbital and intrinsic spin in a basis characterized with quantum numbers relating to $\boldsymbol{J}^2$, J_z, $\boldsymbol{l}^2$ and $\boldsymbol{s}^2$, respectively. Since $J_z = l_z + s_z$, in the above case, a fixed m-eigenvalue will occur but not necessarily a fixed m_l, m_s value. This follows from the fact that $\boldsymbol{J}^2$ does not commute with l_z or with s_z but only with the sum $l_z + s_z$. (The proof of this is left as an exercise.)

We now study the effect of acting with $\boldsymbol{J}^2$ on the state vectors that are eigenvectors of the "uncoupled" ($\boldsymbol{l}^2$, l_z, $\boldsymbol{s}^2$, s_z) basis. Since one can write $\boldsymbol{J}^2$ as

$$\boldsymbol{J}^2 = \boldsymbol{l}^2 + \boldsymbol{s}^2 + 2l_z s_z + l_+ s_- + l_- s_+ \;, \tag{1.74}$$

acting on the vectors $\psi(lm_l,\, \frac{1}{2}m_s)$ one obtains

$$J^2\psi\left(lm_l, \tfrac{1}{2}m_s\right) = \hbar^2\left\{l(l+1) + \tfrac{3}{4} + 2m_l m_s\right\}\psi\left(lm_l, \tfrac{1}{2}m_s\right) + \alpha\psi\left(l\,m_l+1, \tfrac{1}{2}\,m_s-1\right) + \beta\psi\left(l\,m_l-1, \tfrac{1}{2}\,m_s+1\right) . \tag{1.75}$$

From (1.75) it becomes clear that the eigenvectors $\psi(lm_l, \frac{1}{2}m_s)$ are not in general eigenvectors of J^2, although they are eigenvectors of l^2, s^2 and J_z. The right-hand side of (1.75) represents a 2×2 matrix spanned by the configurations $\psi(lm_l, \frac{1}{2}m_s)$ with a fixed value of total magnetic quantum number m ($= m_l + m_s$). By diagonalizing this matrix one obtains two eigenvalues of j, i.e., $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$. Of course, in the two extreme cases (see the problem set) $j = l + \frac{1}{2}$, $m = l + \frac{1}{2}$ and $j = l + \frac{1}{2}$, $m = -l - \frac{1}{2}$, only one component, $\psi(l, m_l = l, \frac{1}{2}, m_s = +\frac{1}{2})$ and $\psi(l, m_l = -l, \frac{1}{2}, m_s = -\frac{1}{2})$, respectively, results.

As an example, we take the case of $l = 4$, $s = \frac{1}{2}$ that can be combined to form both the $j = \frac{9}{2}$ and $j = \frac{7}{2}$ total angular momenta. The states obtained are

$$\begin{aligned}
&\psi\left(l=4,\ s=\tfrac{1}{2},\ j=\tfrac{9}{2},\ m=+\tfrac{9}{2}\right)\\
&\quad= \psi\left(l=4,\ m_l=4,\ s=\tfrac{1}{2},\ m_s=+\tfrac{1}{2}\right) ,\\
&\psi\left(l=4,\ s=\tfrac{1}{2},\ j=\tfrac{9}{2},\ m=+\tfrac{7}{2}\right)\\
&\quad= \alpha\psi\left(l=4,\ m_l=4,\ s=\tfrac{1}{2},\ m_s=-\tfrac{1}{2}\right)\\
&\qquad+ \beta\psi\left(l=4,\ m_l=3,\ s=\tfrac{1}{2},\ m_s=+\tfrac{1}{2}\right) ,\\
&\psi\left(l=4,\ s=\tfrac{1}{2},\ j=\tfrac{7}{2},\ m=+\tfrac{7}{2}\right)\\
&\quad= \beta\psi\left(l=4,\ m_l=4,\ s=\tfrac{1}{2},\ m_s=-\tfrac{1}{2}\right)\\
&\qquad- \alpha\psi\left(l=4,\ m_l=+3,\ s=\tfrac{1}{2},\ m_s=+\tfrac{1}{2}\right) ,\\
&\quad\dots\\
&\psi\left(l=4,\ s=\tfrac{1}{2},\ j=\tfrac{9}{2},\ m=-\tfrac{9}{2}\right)\\
&\quad= \psi\left(l=4,\ m_l=-4,\ s=\tfrac{1}{2},\ m_s=-\tfrac{1}{2}\right) .
\end{aligned} \tag{1.76}$$

In general, the eigenvectors of J^2, J_z, l^2, s^2, denoted by $\psi(ls = \frac{1}{2}, jm)$ can be expanded in the eigenvectors of l^2, l_z, s^2, s_z that are given by $\psi(lm_l, \frac{1}{2}\,m_s)$ as follows:

$$\psi\left(ls=\tfrac{1}{2},\ jm\right) = \sum_{m_l, m_s} \langle lm_l,\ \tfrac{1}{2}m_s | ls=\tfrac{1}{2}, jm\rangle \psi\left(lm_l,\ \tfrac{1}{2}m_s\right) . \tag{1.77}$$

The coefficients that establish the transformation of one complete basis to the other complete basis $\langle lm_l, \frac{1}{2}m_s | ls = \frac{1}{2}, jm\rangle$ are denoted as Clebsch-Gordan

Table 1.1 Analytic expressions for the Clebsch-Gordan coefficients appearing in (1.79) for coupling the orbital angular momentum l with the intrinsic spin $s = 1/2$ to a total angular momentum $j = l \pm 1/2$

j	$m_s = +\frac{1}{2}$	$m_s = -\frac{1}{2}$
$l+\frac{1}{2}$	$\left(\frac{l+1/2+m}{2l+1}\right)^{1/2}$	$\left(\frac{l+1/2-m}{2l+1}\right)^{1/2}$
$l-\frac{1}{2}$	$\left(\frac{l+1/2-m}{2l+1}\right)^{1/2}$	$-\left(\frac{l+1/2+m}{2l+1}\right)^{1/2}$

(Clebsch 1872, Gordan 1875) or vector-coupling coefficients. In the new, "coupled" basis of (1.77) one has the following eigenvalue relations

$$\begin{aligned}
l^2\psi\left(ls=\tfrac{1}{2},\, jm\right) &= \hbar^2 l(l+1)\psi\left(ls=\tfrac{1}{2},\, jm\right)\,,\\
s^2\psi\left(ls=\tfrac{1}{2},\, jm\right) &= \hbar^2 \tfrac{3}{4}\psi\left(ls=\tfrac{1}{2},\, jm\right)\,,\\
J^2\psi\left(ls=\tfrac{1}{2},\, jm\right) &= \hbar^2 j(j+1)\psi\left(ls=\tfrac{1}{2},\, jm\right)\,,\\
J_z\psi\left(ls=\tfrac{1}{2},\, jm\right) &= \hbar m\psi\left(ls=\tfrac{1}{2},\, jm\right)\,.
\end{aligned} \tag{1.78}$$

Written explicitly, the total state vector for a spin $s = \frac{1}{2}$ fermion particle becomes

$$\begin{aligned}
\psi\left(nls=\tfrac{1}{2},\, jm\right) = R_{nl}(r)\Big\{&\langle lm-\tfrac{1}{2},\, \tfrac{1}{2}+\tfrac{1}{2}|l\tfrac{1}{2},\, jm\rangle Y_l^{m-1/2}(\theta,\varphi)\chi_{+1/2}^{1/2}(\boldsymbol{\sigma})\\
&+\langle lm+\tfrac{1}{2},\, \tfrac{1}{2}-\tfrac{1}{2}|l\tfrac{1}{2},\, jm\rangle Y_l^{m+1/2}(\theta,\varphi)\chi_{-1/2}^{1/2}(\boldsymbol{\sigma})\Big\}\,,
\end{aligned} \tag{1.79}$$

with the $\langle \ldots | \ldots \rangle$ Clebsch-Gordan coefficients given in Table 1.1.

1.4 Coupling of Two Angular Momenta: Clebsch-Gordan Coefficients

In this section, we concentrate on the coupling of two distinct angular momenta, but now for the more general case of angular momentum operators $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ that may have an m-projection J_{1z}, J_{2z} representing both integer or half-integer values. In that case, the total angular momentum operator is expressed by the sum $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$. The set of four commuting operators

$$\left\{\boldsymbol{J}_1^2,\, J_{1z},\, \boldsymbol{J}_2^2,\, J_{2z}\right\}\,, \tag{1.80}$$

are characterized by the common eigenvectors, expressed by the product state vectors

$$|j_1m_1, j_2m_2\rangle \equiv |j_1m_1\rangle|j_2m_2\rangle \ . \tag{1.81}$$

The other set of commuting operators

$$\left\{\boldsymbol{J}^2, J_z, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2\right\}, \tag{1.82}$$

can also be characterized by a set of common eigenvectors, being linear combinations of the eigenvectors (1.81), such that they form eigenvectors of $\boldsymbol{J}^2$. One also denotes them as the "coupled" and "uncoupled" eigenvectors that are related via the expression

$$|j_1j_2;jm\rangle = \sum_{\substack{m_1,m_2 \\ (m_1+m_2=m)}} \langle j_1m_1, j_2m_2|j_1j_2, jm\rangle|j_1m_1\rangle|j_2m_2\rangle \ . \tag{1.83}$$

The overlap coefficients $\langle \ldots | \ldots \rangle$, going from one basis to the other are called the Clebsch-Gordan coefficients. To simplify the notation one frequently uses an abbreviation in the ket side of the bracket, i.e.,

$$\langle j_1m_1, j_2m_2|j_1j_2, jm\rangle \rightarrow \langle j_1m_1, j_2m_2|jm\rangle \ . \tag{1.84}$$

In order to determine the relative phases for the Clebsch-Gordan coefficients, we shall use the Condon-Shortley phase convention (Condon, Shortley 1935, Edmonds 1957), which is basically defined as follows.

i) When acting with the ladder operator $J_\pm$ on the eigenvectors $|j_1j_2;jm\rangle$, we define the phase as

$$J_\pm|j_1j_2;jm\rangle = \mathrm{e}^{\mathrm{i}\delta}\,\hbar\big((j \mp m)(j \pm m + 1)\big)^{1/2}|j_1j_2;jm \pm 1\rangle \ , \tag{1.85}$$

with $\mathrm{e}^{\mathrm{i}\delta} = +1$.

ii) By acting with the operator $\boldsymbol{J}^2$ ($\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$) on the states with the extreme projection quantum number $M = j_1 + j_2$ or $M = -j_1 - j_2$, one gets the result that

$$\boldsymbol{J}^2|j_1j_2;jm = \pm(j_1+j_2)\rangle = (j_1+j_2)(j_1+j_2+1)\,\hbar^2|j_1j_2, jm = \pm(j_1+j_2)\rangle \ , \tag{1.86}$$

indicating that the state in (1.86) is an eigenstate of $\boldsymbol{J}^2$ and J_z with eigenvalues $(j_1+j_2)(j_1+j_2+1)\,\hbar^2$ and $M = \pm\,\hbar(j_1+j_2)$, respectively. Thus we get, by applying (1.83) for this particular case

$$\begin{aligned} &|j_1j_2; j = j_1 + j_2,\, m = \pm(j_1+j_2)\rangle \\ &\qquad = \mathrm{e}^{\mathrm{i}\alpha}\,|j_1m_1 = \pm j_1\rangle|j_2, m_2 = \pm j_2\rangle \ , \end{aligned} \tag{1.87}$$

which, with the choice $\mathrm{e}^{\mathrm{i}\alpha} = +1$, gives the aligned Clebsch-Gordan coefficients

$$\langle j_1m_1 = \pm j_1, j_2m_2 = \pm j_2|j = j_1 + j_2,\, m = \pm(j_1+j_2)\rangle = +1 \ . \tag{1.88}$$

iii) By acting now with the ladder (lowering) operator $J_- = J_{1-} + J_{2-}$, on (1.87), and relating that result to the explicit form of the state vector (1.83) with $j = j_1 + j_2$, $m = j_1 + j_2 - 1$; we get

$$J_-|j_1 j_2;\ j = j_1 + j_2,\ m = j_1 + j_2\rangle = \left(J_{1-} + J_{2-}\right)|j_1, m_1 = j_1\rangle|j_2, m_2 = j_2\rangle\ , \tag{1.89}$$

and

$$\begin{aligned}|j_1 j_2; j &= j_1 + j_2,\ m = j_1 + j_2 - 1\rangle \\ &= \langle j_1 j_1 - 1, j_2 j_2 | j = j_1 + j_2,\ m = j_1 + j_2 - 1\rangle |j_1, j_1 - 1\rangle |j_2, j_2\rangle \\ &\quad + \langle j_1 j_1, j_2 j_2 - 1 | j = j_1 + j_2,\ m = j_1 + j_2 - 1\rangle |j_1, j_1\rangle |j_2, j_2 - 1\rangle\ .\end{aligned} \tag{1.90}$$

This leads to the identification

$$\begin{aligned}\langle j_1 j_1 - 1, j_2 j_2 | j = j_1 + j_2,\ m = j_1 + j_2 - 1\rangle &= \left(j_1/(j_1 + j_2)\right)^{1/2} \\ \langle j_1 j_1, j_2 j_2 - 1 | j = j_1 + j_2,\ m = j_1 + j_2 - 1\rangle &= \left(j_2/(j_1 + j_2)\right)^{1/2}\ .\end{aligned} \tag{1.91}$$

With the value of $m = j_1 + j_2 - 1$, another state can be constructed, i.e., $|j = j_1 + j_2 - 1,\ m = j_1 + j_2 - 1\rangle$ which should be constructed from the same uncoupled states that appear in (1.90). By imposing the condition of orthogonality, one can deduce both the absolute value and the relative phase of the Clebsch-Gordan coefficients

$$\begin{aligned}&\langle j_1 j_1 - 1,\ j_2 j_2 | j = j_1 + j_2 - 1,\ m = j_1 + j_2 - 1\rangle\ , \quad \text{and} \\ &\langle j_1 j_1,\ j_2 j_2 - 1 | j = j_1 + j_2 - 1,\ m = j_1 + j_2 - 1\rangle\ .\end{aligned} \tag{1.92}$$

The absolute phases are now defined by the condition that for any given total J and projection M one has

$$\langle JM|J_{1z}|J - 1,\ M\rangle \geq 0\ . \tag{1.93}$$

This condition, written out for the Clebsch-Gordan coefficients making up the states $|JM\rangle$ and $|J - 1, M\rangle$ becomes

$$\sum_{m_1, m_2} m_1 \langle j_1 m_1, j_2 m_2 | JM\rangle \langle j_1 m_1, j_2 m_2 | J - 1M\rangle \geq 0\ . \tag{1.94}$$

The above condition (1.94) can be shown to be equivalent to the condition (Brussaard 1967)

$$\langle j_1 j_1, j_2 J - j_1 | J, M = J\rangle \geq 0 \qquad \text{for each } J\ . \tag{1.95}$$

These are now the phase conditions of (i), (ii) and (iii) that uniquely define the Clebsch-Gordan coefficients and thus also the coupled state vectors of (1.83).

1.5 Properties of Clebsch-Gordan Coefficients

Since the Clebsch-Gordan coefficients serve as expansion coefficients for a given eigenvector in a specified ortho-normal basis, there exist orthogonality relations that are given by

$$\sum_{m_1,m_2} \langle j_1m_1, j_2m_2|jm\rangle\langle j_1m_1, j_2m_2|j'm'\rangle = \delta_{jj'}\delta_{mm'} \;, \quad \text{and}$$
$$\sum_{j,m}\langle j_1m_1, j_2m_2|jm\rangle\langle j_1m_1', j_2m_2'|jm\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'} \;. \tag{1.96}$$

Interesting symmetry relations exist when interchanging the two angular momenta that become coupled, e.g., (de-Shalit, Talmi 1963)

$$\langle j_1m_1, j_2m_2|jm\rangle = (-1)^{j_1+j_2-j}\langle j_2m_2, j_1m_1|jm\rangle \;. \tag{1.97}$$

In interchanging either j_1 and j or j_2 and j, more complex relations result, since angular momenta are coupled in a certain 'direction' e.g. j_1 with j_2 to form the angular momentum j and in that order. More symmetric ways of coupling can be made.

Two angular momentum states $|jm\rangle$ can be coupled to a total angular momentum of zero. The normalized state then becomes

$$|\Phi_0\rangle = \sum_m (2j+1)^{-1/2}(-1)^{j-m}|j,m\rangle|j,-m\rangle \;. \tag{1.98}$$

Using the Wigner $1j$-symbol (Brussaard 1967, Wigner 1959)

$$\begin{pmatrix} j \\ mm' \end{pmatrix} = (-1)^{j+m}\delta_{m,-m'} = (-1)^{j-m'}\delta_{m,-m'} \;, \tag{1.99}$$

it follows that the combination

$$(2j+1)^{1/2}|\Phi_0\rangle = \sum_{m_1,m_2} \begin{pmatrix} j \\ m_1m_2 \end{pmatrix} |jm_1\rangle|jm_2\rangle \;, \tag{1.100}$$

forms an angular momentum invariant.

Using the same method, one can construct out of the three subsystems j_1, j_2 and j_3 a system with total angular momentum zero. The state vector thus constructed becomes

$$|\Psi_0\rangle = \sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle \;, \tag{1.101}$$

with the coefficients $\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$, the Wigner $3j$-symbols (Wigner 1959, de-Shalit and Talmi 1963, Brussaard 1967). By constructing the state $|\Psi_0\rangle$ by first coupling the individual angular momenta j_1 and j_2 to an angular momentum j_3 and then subsequent to the third angular momentum j_3 to form a state of total angular momentum zero, a relation between the Wigner $3j$-symbol and the Clebsch-Gordan coefficients is obtained. This relation is given by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \langle j_1 m_1, j_2 m_2 | j_3 \; -m_3 \rangle \; , \tag{1.102}$$

symmetry properties under the interchange of any two angular momenta of the set (j_1, j_2, j_3) become very simple:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \\ &= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} \\ &= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} = \dots \; , \end{aligned} \tag{1.103}$$

or, a phase factor +1 for an *even* permutation and a phase factor $(-1)^{j_1+j_2+j_3}$ for an *odd* permutation. Moreover, one gets the relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \; . \tag{1.104}$$

The former orthogonality relations (1.96) for the Clebsch-Gordan coefficients now rewritten in terms of the Wigner $3j$-symbols become

$$\sum_{m_1,m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{1}{2j_3+1} \delta_{j_3 j_3'} \delta_{m_3 m_3'} \; , \tag{1.105}$$

$$\sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1 \; , \tag{1.106}$$

$$\sum_{j_3,m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \; . \tag{1.107}$$

Extensive sets of tables of Wigner $3j$-symbols exist (e.g. Rotenberg et al. 1959).

Explicit calculations of the Wigner $3j$-symbol are easily performed using the expression (de-Shalit, Talmi 1963).

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \delta_{m_1+m_2,-m_3} \big(((j_1+j_2-j_3)!\,(j_2+j_3-j_1)! \\ &\times (j_3+j_1-j_2)!) / (j_1+j_2+j_3+1)! \big)^{1/2} \\ &\times \big((j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)! \big)^{1/2} \\ &\times \sum_t (-1)^{j_1-j_2-m_3+t} \big(t!(j_1+j_2-j_3-t)!(j_3-j_2+m_1+t)! \\ &\times (j_3-j_1-m_2+t)!(j_1-m_1-t)!(j_2+m_2-t)! \big)^{-1} \; . \end{aligned} \tag{1.108}$$

with $(-m)! = 0$ if m is positive, t is entire and $0! = 1$, so the following conditions hold

$$\begin{aligned}
&t \geq 0 \\
&j_1 + j_2 - j_3 \geq t \\
&-j_3 + j_2 - m_1 \leq t \\
&-j_3 + j_1 + m_2 \leq t \\
&j_1 - m_1 \geq t \\
&j_2 + m_2 \geq t\,.
\end{aligned} \tag{1.109}$$

In Chap. 9, a FORTRAN program is given that evaluates (1.108) numerically. As an example, we evaluate the $3j$-symbol

$$\begin{pmatrix} j & 1 & j \\ -m & 0 & m \end{pmatrix} .$$

The conditions (1.109) give the restrictions on t

$$\begin{aligned}
&t \geq 0\,;\ 1 \geq t\,;\ -j + 1 + m \leq t\,;\ 0 \leq t\,; \\
&j + m \geq t\,;\ 1 \geq t \quad \text{or} \quad t = 0, 1\,.
\end{aligned}$$

Calculating in detail, one gets

$$\begin{aligned}
\begin{pmatrix} j & 1 & j \\ -m & 0 & m \end{pmatrix} &= \big((2j-1)!/(2j+2)!\big)^{1/2} \\
&\times \big((j-m)!(j+m)!(j+m)!(j-m)!\big)^{1/2} \\
&\times (-1)^{j-1-m}\Big[\big((j-1-m)!(j+m)!\big)^{-1} - \big((j-m)!(j+m-1)!\big)^{-1}\Big] \\
&= (-1)^{j-m} m/\big(j(j+1)(2j+1)\big)^{1/2}\,.
\end{aligned}$$

1.6 Racah Recoupling Coefficients: Coupling of Three Angular Momenta

In the case of a system described by three independent angular momentum operators $\boldsymbol{J}_1$, $\boldsymbol{J}_2$, $\boldsymbol{J}_3$; one can again form the total angular momentum operator $\boldsymbol{J}$ defined as

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3\,. \tag{1.110}$$

The six commuting operators

$$\left\{\boldsymbol{J}_1^2, J_{1z}, \boldsymbol{J}_2^2, J_{2z}, \boldsymbol{J}_3^2, J_{3z}\right\}, \tag{1.111}$$

have a set of common eigenvectors, the product vectors

$$|j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle\,. \tag{1.112}$$

For the three angular momentum operators, it is now possible to form three sets of commuting operators:

$$\left\{\boldsymbol{J}^2, J_z, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_{12}^2\right\}, \tag{1.113}$$

$$\left\{\boldsymbol{J}^2, J_z, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_{23}^2\right\}, \tag{1.114}$$

$$\left\{\boldsymbol{J}^2, J_z, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_{13}^2\right\}, \tag{1.115}$$

with the eigenvectors

$$|(j_1 j_2) J_{12} j_3; JM\rangle , \tag{1.116}$$

$$|j_1 (j_2 j_3) J_{23}; JM\rangle , \tag{1.117}$$

$$|(j_1 j_3) J_{13} j_2; JM\rangle , \tag{1.118}$$

respectively.

It is possible to use a diagrammatic way of expressing the vector coupled eigenstates (Brussaard, Glaudemans 1977), by using lines and arrows for a given angular momentum and the order in which they are coupled. The intermediate angular momentum is shown by the dashed line (Fig. 1.3).

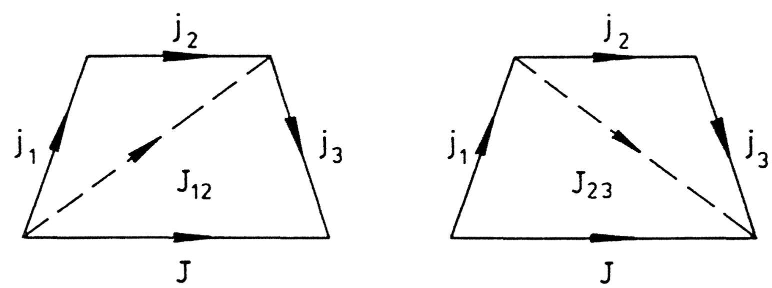


Fig. 1.3. Graphical illustration of two possible ways to construct the angular momentum wave functions for a system where *three* angular momenta are used, according to (1.116) and (1.117). The angular momenta are represented by vectors, the intermediate momenta by dashed-line vectors

Between the three equivalent sets of eigenvectors of (1.116–118), transformations that change from one basis to another can be constructed. We can formally write for such a transformation (de-Shalit, Talmi 1963)

$$|j_1 (j_2 j_3) J_{23}; JM\rangle = \sum_{J_{12}} \langle (j_1 j_2) J_{12} j_3; J | j_1 (j_2 j_3) J_{23}; J\rangle \times |(j_1 j_2) J_{12} j_3; JM\rangle . \tag{1.119}$$

It can easily be shown that the transformation coefficients in (1.119) and in similar relations do not depend on the projection quantum number M. Now by explicitly carrying out the recoupling from the states $|j_1(j_2 j_3)J_{23}; JM\rangle$ to the coupling scheme $|(j_1 j_2)J_{12} j_3; JM\rangle$ (Appendix B), one obtains the detailed form of the recoupling coefficient of (1.119). In this particular situation, a full sum over all magnetic quantum numbers of products of *four* Wigner $3j$-symbols results. The latter, defined as an angular momentum invariant quantity (no longer dependent on the specific orientation of a quantization axis), the Wigner $6j$-symbol, leads to the following result (Wigner 1959, Brussaard 1967)

$$\langle j_1(j_2j_3)J_{23};J|(j_1j_2)J_{12}j_3;J\rangle = (-1)^{j_1+j_2+j_3+J}\hat{J}_{12}\hat{J}_{23}\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix}, \tag{1.120}$$

(using the notation $\hat{J} \equiv (2J+1)^{1/2}$).

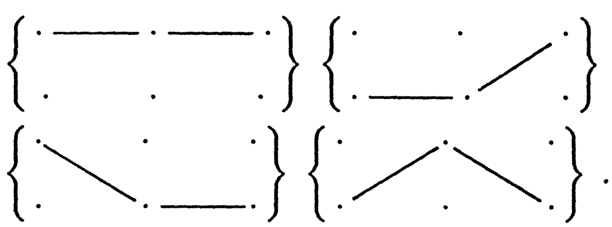
The precise definition of the $6j$-symbol in terms of the $3j$-symbols reads (de-Shalit, Talmi 1963)

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \sum_{\text{all } m_i, m_i'} (-1)^{\Sigma j_i + \Sigma l_i + \Sigma m_i + \Sigma m_i'} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \times \begin{pmatrix} j_1 & l_2 & l_3 \\ -m_1 & m_2' & -m_3' \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -m_1' & -m_2 & m_3' \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ m_1' & -m_2' & -m_3 \end{pmatrix} \tag{1.121}$$

and very much resembles a "contraction of tensors" (one sums over projection quantum numbers $m_1, m_2, \ldots, m_3'$, both of which always show up in different $3j$-symbols with opposite sign). We show in Chap. 2 that, indeed, the $6j$-symbol is a full contraction not on cartesian but on spherical tensors (Wigner 1959).

1.7 Symmetry Properties of $6j$-Symbols

Because of the very structure of the definition in (1.121), in each $6j$-symbol four angular momentum couplings have to be satisfied in order to be non-vanishing. In shorthand notation, replacing the angular momenta with dots, one has the couplings



(1.122)

Here we quote some often used symmetry properties. A more detailed account can be found in various texts (de-Shalit, Talmi 1963, Edmonds 1957, Rose, Brink 1967, Brussaard 1967, Brink, Satchler 1962)

i) $$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = 0, \tag{1.123}$$

unless the triangular (coupling) conditions $(j_1j_2j_3)$, $(j_1l_2l_3)$, $(l_1l_2j_3)$, $(l_1j_2l_3)$ are fulfilled

ii) $$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_3 & j_1 \\ l_2 & l_3 & l_1 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_2 \\ l_1 & l_3 & l_2 \end{Bmatrix}$$
$$= \begin{Bmatrix} l_1 & l_2 & j_3 \\ j_1 & j_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_1 & l_2 & l_3 \\ l_1 & j_2 & j_3 \end{Bmatrix} = \dots , \tag{1.124}$$

iii) orthogonality relation

$$\sum_j (2j+1) \begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j'' \end{Bmatrix} = \delta_{j'j''}(2j'+1)^{-1} , \tag{1.125}$$

iv) special case

$$\begin{Bmatrix} j_1 & j_2' & j_3 \\ j_2 & j_1' & 0 \end{Bmatrix} = (-1)^{j_1+j_2+j_3} \left(\hat{j}_1 \hat{j}_2\right)^{-1} \delta_{j_1 j_1'} \delta_{j_2 j_2'} , \tag{1.126}$$

v) Explicit form: Racah formula (de-Shalit, Talmi 1963)

$$\begin{aligned} \begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} &= \Delta(j_1 j_2 j_3) \Delta(j_1 l_2 l_3) \Delta(l_1 j_2 l_3) \Delta(l_1 l_2 j_3) \\ &\times \sum_t (-1)^t (t+1)! \big[(t - j_1 - j_2 - j_3)! (t - j_1 - l_2 - l_3)! \\ &\times (t - l_1 - j_2 - l_3)! (t - l_1 - l_2 - j_3)! (j_1 + j_2 + l_1 + l_2 - t)! \\ &\times (j_2 + j_3 + l_2 + l_3 - t)! (j_3 + j_1 + l_3 + l_1 - t)! \big]^{-1} , \end{aligned} \tag{1.127}$$

with

$$\Delta(abc) = \left[(a+b-c)!(b+c-a)!(c+a-b)!/(a+b+c+1)! \right]^{1/2} , \tag{1.128}$$

and the condition of having non-negative values of the integer in the factorial expression in (1.127).

1.8 Wigner $9j$-Symbols: Coupling and Recoupling of Four Angular Momenta

Similarly to the methods used in Sect. 1.6, we can construct the total angular momentum operator corresponding to the sum of the four independent angular momentum operators as

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3 + \boldsymbol{J}_4 . \tag{1.129}$$

In constructing the total set of commuting operators one has in the uncoupled representation,

$$\left\{ \boldsymbol{J}_1^2, J_{1z}, \boldsymbol{J}_2^2, J_{2z}, \boldsymbol{J}_3^2, J_{3z}, \boldsymbol{J}_4^2, J_{4z} \right\} , \tag{1.130}$$

which have as eigenvectors the product vectors

$$|j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle |j_4 m_4\rangle . \tag{1.131}$$

In the coupled representation, one needs two intermediate angular momentum operators for which a large choice exists. Coupling pairwise, one has three possibilities

$$\begin{aligned}
&\boldsymbol{J}^2, J_z, \boldsymbol{J}_{12}^2, \boldsymbol{J}_{34}^2, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_4^2\,, \\
&\boldsymbol{J}^2, J_z, \boldsymbol{J}_{13}^2, \boldsymbol{J}_{24}^2, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_4^2\,, \\
&\boldsymbol{J}^2, J_z, \boldsymbol{J}_{14}^2, \boldsymbol{J}_{23}^2, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_4^2\,,
\end{aligned} \tag{1.132}$$

(Fig. 1.4), with eigenvectors

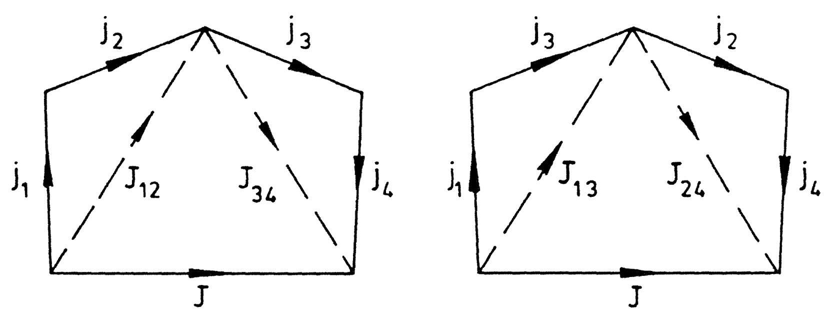
$$\begin{aligned}
&|(j_1 j_2) J_{12} (j_3 j_4) J_{34}; JM\rangle\,, \\
&|(j_1 j_3) J_{13} (j_2 j_4) J_{24}; JM\rangle\,, \\
&|(j_1 j_4) J_{14} (j_2 j_3) J_{23}; JM\rangle\,,
\end{aligned} \tag{1.133}$$

respectively.

There exist other possibilities, too, however, such as

$$\boldsymbol{J}^2, J_z, \boldsymbol{J}_{12}^2, \boldsymbol{J}_{123}^2, \boldsymbol{J}_1^2, \boldsymbol{J}_2^2, \boldsymbol{J}_3^2, \boldsymbol{J}_4^2\,, \tag{1.134}$$

shown in Fig. 1.4. The latter method is probably the best adapted to extend coupling to n angular momenta by successive coupling of an extra angular momentum to the former $n-1$ system (Yutsis et al. 1962). Here, too, many possible recoupling schemes and recoupling coefficients can be obtained (Edmonds 1957). Here we only discuss recoupling between the states of (1.133) since they lead to the Wigner $9j$-symbol, e.g., (de-Shalit, Talmi 1963)



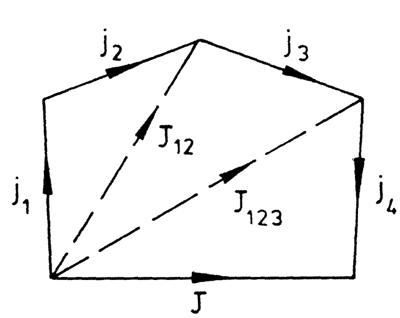


Fig. 1.4. Graphical illustration of possible ways to construct the angular momentum wave functions for a system where *four* angular momenta are used, according to (1.133). In the lower part, we present the more general way of constructing the four-angular momentum system by specifying J_{12}, J_{123} as intermediate angular momenta, respectively

$$|(j_1j_3)J_{13}(j_2j_4)J_{24};JM\rangle = \sum_{J_{12},J_{34}} \hat{J}_{13}\hat{J}_{24}\hat{J}_{12}\hat{J}_{34} \times \left\{ \begin{matrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{matrix} \right\} |(j_1j_2)J_{12}(j_3j_4)J_{34};JM\rangle \ . \tag{1.135}$$

Precise definitions of the Wigner $9j$-symbol as a full contraction over products of 6 $3j$-symbols can be found (Edmonds 1957, de-Shalit, Talmi 1963). In the present context, where we shall concentrate on the nuclear shell model, we quote a special case that often occurs:

$$\left\{ \begin{matrix} j_1 & j_2 & J \\ j_4 & j_3 & k \end{matrix} \right\} = (-1)^{j_2+J+j_3+k} \hat{J}\hat{k} \left\{ \begin{matrix} j_1 & j_2 & J \\ j_3 & j_4 & J \\ k & k & 0 \end{matrix} \right\} \ . \tag{1.136}$$

Also, we point out that the general expression of (1.135) can be used when recoupling from a $(j\,j)$ coupling basis into an (LS) coupling basis if we consider cases with two fermions. Thus one can relate the states $|(l_1l_2)L(\frac{1}{2}\,\frac{1}{2})S;JM\rangle$ and $|(l_1\frac{1}{2})j_1(l_2\frac{1}{2})j_2;JM\rangle$ by the transformation

$$|(l_1l_2)L\left(\tfrac{1}{2}\,\tfrac{1}{2}\right)S;JM\rangle = \sum_{j_1,j_2} \hat{L}\hat{S}\hat{j}_1\hat{j}_2 \left\{ \begin{matrix} l_1 & l_2 & L \\ \frac{1}{2} & \frac{1}{2} & S \\ j_1 & j_2 & J \end{matrix} \right\} \times |\left(l_1\tfrac{1}{2}\right)j_1\left(l_2\tfrac{1}{2}\right)j_2;JM\rangle \ , \tag{1.137}$$

a relation that gives the $(jj) \rightarrow (LS)$ basis transformation.

1.9 Classical Limit of Wigner $3j$-Symbols

It is now possible to construct a classical (in the limit of large angular momenta) model (Brussaard, Tolhoek 1957, Brussaard 1967) for angular momentum coupling and thus also for the Wigner $3j$- (and similarly for the $6j$-, $9j$-, $3nj$-) symbol. We make use of the fact that in quantum mechanics it is only possible to specify both the length and the projection on a quantization axis of the angular momentum. Therefore, a precessing vector model results where for constant precession velocity the azimuthal angle has a constant probability distribution. Since the Clebsch-Gordan coefficients denote the expansion coefficients in an orthonormal basis, the square can be interpreted as a probability. Thus, for the uncoupled representation where $\boldsymbol{J}_1^2$, J_{1z}, $\boldsymbol{J}_2^2$ and J_{2z} are the commuting operators, the coefficients $|\langle j_1m_1, j_2m_2|jm\rangle|^2$ denote the probability that in a state with fixed (j_1m_1) and (j_2m_2), a given value of (j,m) will result with j expressing the length of the angular momentum vector (correct only for large values of j), (Fig. 1.5). Similarly, $|\langle j_1m_1j_2m_2|jm\rangle|^2$ (Fig. 1.5) can be interpreted, for the coupled basis where eigenstates of the operators $\boldsymbol{J}^2$, J_z, $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$ are considered, as the probability that for given (j,m) the values m_1 and m_2 will result as projection quantum numbers relating to the angular momenta j_1 and j_2, respectively. One can even calculate this

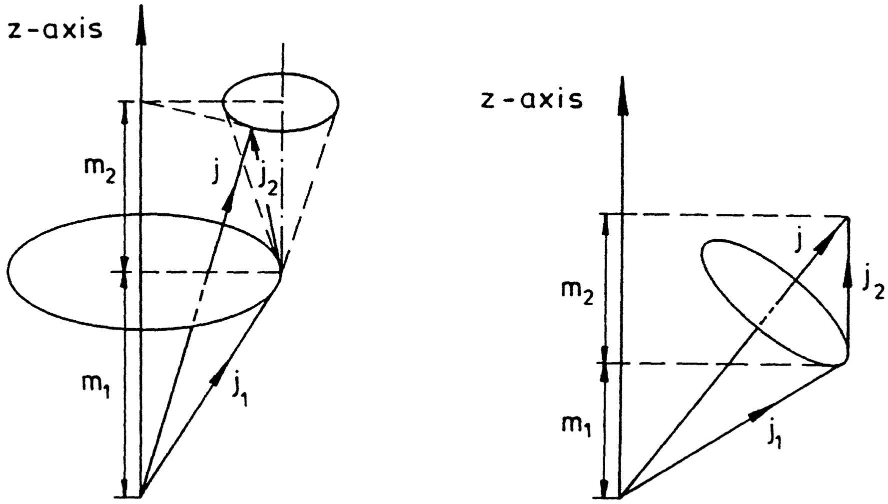


Fig. 1.5. Graphical representation of two angular momenta $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$, shown as vectors that make a precession around the z-axis with constant angular velocity (vector model). Using the addition to a momentum $\boldsymbol{j} = \boldsymbol{j}_1 + \boldsymbol{j}_2$, the probability of obtaining a given value for the length j, given fixed m_1 and m_2 values, is given by the Clebsch-Gordan coefficient squared $|\langle j_1 m_1, j_2 m_2 | jm >|^2$. If the two vectors $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ are coupled to form the total angular momentum $\boldsymbol{j}$ (which is a constant of motion), the two vectors will make a precession around the direction of $\boldsymbol{j}$. For fixed value of the length of $\boldsymbol{j}$ and projection m, the projections m_1 and m_2 can be obtained again as a probability distribution given by the Clebsch-Gordan coefficient squared $| < j_1 m_1, j_2 m_2 | jm > |^2$

distribution in both cases from probability considerations (Edmonds 1957 gives an explicit calculation). Extending the above arguments, classical models can also be constructed for interpreting higher $3n - j$ symbols (Brussaard 1967).

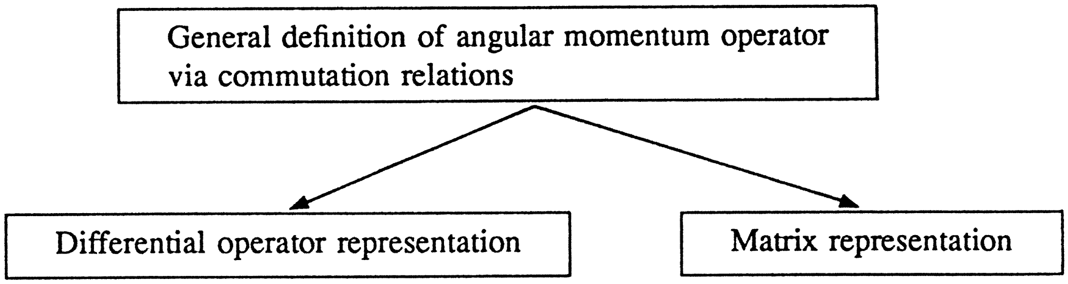
Short Overview of Angular Momentum Coupling Formulas

One-particle central force motion-orbital angular momentum

l_x, l_y, l_z : differential operators

$$
\begin{aligned}
\left[l^2, l_i\right] &= 0 \\
\left[l_i, l_j\right] &= \mathrm{i}\varepsilon_{ijk}\, \hbar l_k \\
\left[l_+, l_-\right] &= 2\,\hbar l_z \\
\left[l_z, l_+\right] &= \hbar l_+ \\
\left[l_z, l_-\right] &= -\,\hbar l_-
\end{aligned}
$$

$$
l_\pm |lm\rangle = \hbar\big(l(l+1) - m(m \pm 1)\big)^{1/2} |l, m \pm 1\rangle \; .
$$



$l = 0,\ 1,\ 2,\ \ldots$

$j = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ \ldots$

Total angular momentum

$$\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$$

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$$

$$\left\{\boldsymbol{J}_1^2,\ J_{1z},\ \boldsymbol{J}_2^2, J_{2z}\right\} \quad \text{and} \quad \left\{\boldsymbol{J}^2,\ J_z,\ \boldsymbol{J}_1^2,\ \boldsymbol{J}_2^2\right\}$$

$$|j_1 j_2; jm\rangle = \sum_{m_1, m_2} \langle j_1 m_1, j_2 m_2 | jm\rangle |j_1 m_1\rangle |j_2 m_2\rangle$$

$$|j_1 m_1\rangle |j_2 m_2 > = \sum_{j,m} \langle j_1 m_1, j_2 m_2 | jm\rangle |j_1 j_2; jm\rangle \ .$$

Three angular momentum systems

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3$$

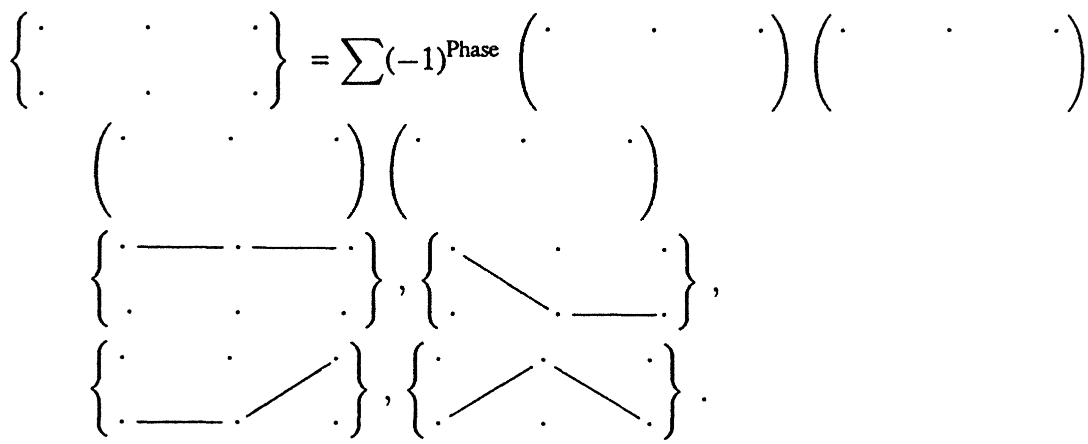
$$\left\{\boldsymbol{J}_1^2,\ J_{1z},\ \boldsymbol{J}_2^2,\ J_{2z}, \boldsymbol{J}_3^2,\ J_{3z}\right\} \to |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle$$

$$\left\{\boldsymbol{J}^2,\ J_z,\ \boldsymbol{J}_1^2,\ \boldsymbol{J}_2^2,\ \boldsymbol{J}_3^2,\ \boldsymbol{J}_{12}^2\right\} \to |(j_1 j_2) J_{12} j_3; JM\rangle$$

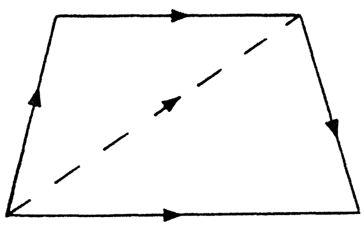
$$\left\{\boldsymbol{J}^2,\ J_z,\ \boldsymbol{J}_1^2,\ \boldsymbol{J}_2^2,\ \boldsymbol{J}_3^2,\ \boldsymbol{J}_{13}^2\right\} \to |(j_1 j_3) J_{13} j_2; JM\rangle$$

$$\left\{\boldsymbol{J}^2,\ J_z,\ \boldsymbol{J}_1^2,\ \boldsymbol{J}_2^2,\ \boldsymbol{J}_3^2,\ \boldsymbol{J}_{23}^2\right\} \to |j_1 (j_2 j_3) J_{23}; JM\rangle \ .$$

Recoupling Wigner 6j-symbol



Notation:



Four angular momentum systems

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3 + \boldsymbol{J}_4$$

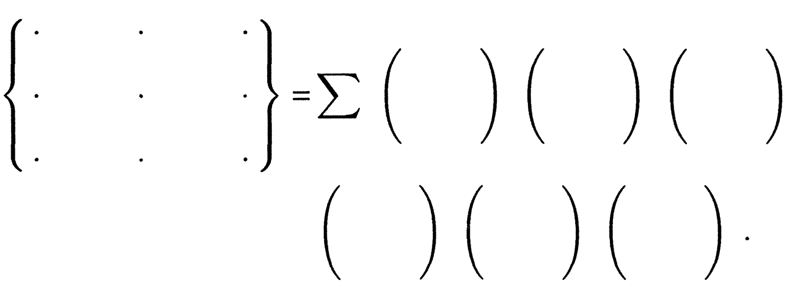
$$\left\{\boldsymbol{J}_1^2,\ J_{1z},\ \boldsymbol{J}_2^2,\ J_{2z},\ \boldsymbol{J}_3^2,\ J_{3z},\ \boldsymbol{J}_4^2,\ J_{4z}\right\}$$

Basis states $\longrightarrow |j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle|j_4m_4\rangle$

$$\left\{\boldsymbol{J}^2,\ J_z,\ \boldsymbol{J}_1^2,\ \boldsymbol{J}_2^2,\ \boldsymbol{J}_3^2,\ \hat{\boldsymbol{J}}_4^2\ \begin{matrix}\text{two intermediate}\\ \text{angular momenta}\end{matrix}\right\}\ \text{e.g. } \boldsymbol{J}_{12}^2,\ \boldsymbol{J}_{34}^2$$

Basis states $\longrightarrow |(j_1j_2)J_{12}(j_3j_4)J_{34};JM\rangle$.

Recoupling-Wigner 9j-symbol



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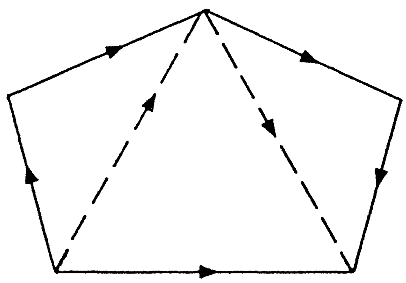

Problems

1.1 Prove the relation that the square of the angular momentum operator $\boldsymbol{l}^2$ can be written as (see (1.10, 11, 12))

$$\boldsymbol{l}^2 = \boldsymbol{r}^2\boldsymbol{p}^2 + \hbar^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) .$$

1.2 Show that the operator $\boldsymbol{J} = \boldsymbol{l} + \boldsymbol{s}$, where $\boldsymbol{l}$ describes the orbital angular momentum operator and $\boldsymbol{s}$ the intrinsic spin angular momentum operator for a spin 1/2 particle, constitutes an angular momentum operator.

1.3 Derive an explicit form of the $2P_{3/2}, m = +1/2$ wave function in terms of the spin and orbital angular momentum wave functions.

1.4 Show that the total angular momentum operator $\boldsymbol{J}^2$ for a nucleon (obtained) by coupling the orbital and intrinsic spin angular momentum operators can be diagonalized in the basis $|lm_l\rangle|1/2m_s\rangle$. Determine the eigenvalues and show that the corresponding eigenfunctions are also eigenfunctions of the Hamiltonian $H = H_0 + a\boldsymbol{l}\cdot\boldsymbol{s}$ with

$$H_0 = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2r^2 .$$

1.5 Prove the orthogonality relations for the Clebsch-Gordan coefficients (see (1.96)).

1.6 Show that the recoupling of (1.135) indeed leads to a Wigner $9-j$ symbol (with no *extra* phase factor).

1.7 Show that the recoupling coefficients $\langle(j_1j_2)J_{12}j_3; JM|j_1(j_2j_3)J_{23}; JM\rangle$, which describe the transformation between states with different coupling order in systems with three angular momenta, are independent of M.

1.8 Determine the relative weights for the $S = 0$ and $S = 1$ intrinsic spin components in the $|(1d_{5/2})^2; J = 2\rangle$ two particle wave function.

1.9 Discuss the classical limit of the Wigner $6-j$ symbols, according to the methods of Sect. 1.9.

*1.10 Calculate the probability density $P(j)$ (i.e. the probability that the length of $\boldsymbol{j}$ lies between j and $j + dj$ is $P(j)dj$) if we suppose that, according to the upper part of Fig. 1.5, $\boldsymbol{j}_1$ rotates at a constant rate about the z-axis with respect to $\boldsymbol{j}_2$. $P(j)$ is then inversely proportional to dj/dt.

1.11 Prove the relation between the Wigner $3j$-symbol and the corresponding Clebsch-Gordan coefficient, as expressed in equation 1.102.

1.12 Determine the matrix representation of the angular momentum operators $\boldsymbol{J}_x$, $\boldsymbol{J}_y$ and $\boldsymbol{J}_z$ for angular momentum 3/2.

*1.13 Discuss, according to the method outlined in Sect. 1.9, the classical limit for the Wigner $6j$-symbol. Construct the graphical representation similar to Fig. 1.5 for the coupling of two angular momenta.